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The Proof of Quantum and Fuzzy Measures as Generalization of Measure That Does Not Generalize Each Other

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Abstract

The studies on quantum and fuzzy theories by Planck and Zadeh, respectively, still continue presently. Based on the mathematical side, these two theories that directly related and become the basis for various studies, both theoretical and applied, are quantum and fuzzy measures. Although in the literature, these are measure generalizations but not substantiated by definition; therefore, the substance does not appear directly. Furthermore, there is also no discussion of the relationship between quantum and fuzzy measures on Boolean σ– algebra. This study accomplishes a proof based on the definition that both the quantum and the fuzzy measures are measure generalizations or do not reciprocally generalize; hence, the measure is the intersection of the two.

Introduction

Light radiation was observed to come from small quanta that can be measured and not from continuous energy waves [1], [2], [3] since Max Planck used the term quantum to observe black body radiation in 1900. This developed into quantum mechanics, where researchers such as Albert Einstein, De Broglie, and Schrodinger, who established special and general relativity theories, wave dualism, and particle wave functions, respectively, were interested. Furthermore, experiments on quantum-related measure theory were developed until 1973, when Gudder [4] reformulated classical measure theory is necessary if the theory is to accurately describe measurements of physical phenomena (generalized measure theory). In addition, Sorkin [5] formulated quantum mechanics as a measure theory in 1994. He demonstrated that classical physics is a special case of quantum physics, by relating quantum mechanics to the additive property of probability in the measure space. In addition, Gudder [6], [7] referred to Sorkin when discussing quantum measure theory as a generalization of measure theory. This generalization not only generalized of the measure theory in known measure spaces but can also invalidate well-known theorems, such as the fundamental theorem of calculus and the Radon-Nikodym theorem.

Zadeh introduced the fuzzy set in 1965 [8] and re-published an article on the probability measure of fuzzy events in 1968 [9]. The studies on fuzzy measure theory have attracted attention, for example, in [10], [11], [12], where fuzzy measure is a generalization of measure. This results in a generalized measure that includes the quantum and fuzzy measures. Furthermore, the fuzzy measure theory is the basis for the application of several models, such as decision-making on uncertain multi-criteria problems [13], fuzzy c-means (FCM) method [14] and fuzzy learning quantization method [15] as clustering methods, and modified generalized Dunn's indices that can be used for the dynamic evaluation of an evolving (including the fuzzy clustering method) structure in streaming data [16].

Anatolij conducted a study in 1988 [17] on the phenomenon of quantum mechanics (quantum probability space) with fuzzy set theory, where the membership function in the set [18] was referred to as fuzzy soft algebra, and the study [17] was named fuzzy quantum spaces. In addition, Duris *et al.*, in 2021 [19], referred to [17], [18] and studied several limit theorems

2. *If $\{B_i\}_{i=1}^{\infty}$ is a collection of descending measurable sets and $\mu(B_1) < \infty$ then:*

$$\mu\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{i\to\infty} \mu(B_i). \quad (2)$$

Results

This study proves the quantum and fuzzy measures as a generalization measure on Boolean σ–algebra based on the definition grouped in the following discussion:

1. Proof of Quantum and Fuzzy Measures as Generalization of Measure
2. Proof that Quantum Measure is not Generalization of Fuzzy Measure
3. Proof that Fuzzy Measure is not Generalization of Quantum Measure.

They are discussed as follows:

Proof of quantum and fuzzy measures as measure generalization

1. Proof of quantum measure as a measure generalization

If (X, M, μ) is the measure space, the proof is conducted by showing that μ is a quantum measure, namely continuous (a), grade-2 additive (b), and regular (c).

a. Continue

According to Theorem 1, μ is continuous.

b. Grade-2 additive

$$\mu(A \dot\cup B \dot\cup C) = \mu(A) + \mu(B) + \mu(C) \quad (3)$$

$$\begin{aligned}&\mu(A \dot\cup B) + \mu(A \dot\cup C) + \mu(B \dot\cup C) - \mu(A) - \mu(B) - \mu(C)\\ &= \mu(A) + \mu(B) + \mu(A) + \mu(C) + \mu(B) + \mu(C) - \mu(A)\\ &-\mu(B) - \mu(C) \ = \mu(A) + \mu(B) + \mu(C)\end{aligned} \quad (4)$$

According to (3) and (4),μ is grade-2 additive.

c. Regular

If $\mu(A) = 0$, hence,

$$\mu(A \dot\cup B) = \mu(A) + \mu(B) = 0 + \mu(B) = \mu(B)$$

If, $\mu(A \dot\cup B) = 0$ therefore,

$\mu(A \dot\cup B) = \mu(A) + \mu(B) = 0$, with $\mu(A) \geq 0$, and $\mu(B) \geq 0$, then $\mu(A) = \mu(B) = 0$, hence, μ is regular.

Therefore, μ is a quantum measure.

2. Proof of Fuzzy measure as a Measure Generalization

If (X, M, μ) is the measure space, the proof is carried out by demonstrating that μ is a fuzzy measure, namely, continuous (a), empty (b), and monotone (c).

a. Continuous

Based on Theorem 1, μ is continuous.

b. Empty

According to the measure definition, it fulfills $\mu(\varnothing) = 0$.

c. Monotone

If $A \subset B$. The finite additive is a special case of the additive countable, which is obtained by taking the last few sets of the countable joint operation as Ø. Consequently, when $A \subset B$, then $\mu(B) = \mu(A) + \mu(B \sim A)$, hence, $\mu(A) \leq \mu(B)$.

Therefore, μ is a fuzzy measure.

Proof of quantum measure not fuzzy measure generalizations

This is carried out with the following counter example, where $X = [0,1]$, M is the σ- algebra of X and ν is the Lebesgue measure constrained to $[0,1]$. For $E \in M$, μ measure is defined as:

$$\mu(E) = \nu(E) - 2\nu\left(\left\{x \in E : x + \frac{3}{4} \in E\right\}\right)$$

$$= \nu(E) - 2\nu\left(E \bigcap \left(E - \frac{3}{4}\right)\right)$$

Then (X, M, μ) is a quantum measure space, which fulfills continuous (1), grade-2 additive (2), and regular (3), but it is not a fuzzy measure (4).

1. Continuous

If A_i sequence ascends in M then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \nu\left(\bigcup_{i=1}^{\infty} A_i\right) - 2\nu\left(\left(\bigcup_{i=1}^{\infty} A_i\right)\bigcap\left(\bigcup_{i=1}^{\infty} A_i - \frac{3}{4}\right)\right)$$

$$= \nu\left(\bigcup_{i=1}^{\infty} A_i\right) - 2\nu\left(\bigcup_{i=1}^{\infty}\left(A_i \cup \left(A_i - \frac{3}{4}\right)\right)\right)$$

$$= \lim_{i\to\infty} \nu(A_i) - 2\lim_{i\to\infty} \nu\left(A_i \bigcap \left(A_i - \frac{3}{4}\right)\right)$$ based on equation (1)

$$= \lim_{i\to\infty}\left(\nu(A_i) - 2\nu\left(A_i \bigcap \left(A_i - \frac{3}{4}\right)\right)\right)$$

$$= \lim_{i\to\infty} \mu(A_i).$$

Furthermore, if the B_i sequence descends at M and it is known that $\mu(B_1) < 1 < \infty$, then

$$\mu\left(\bigcap_{i=1}^{\infty} B_i\right) = \nu\left(\bigcap_{i=1}^{\infty} B_i\right) - 2\nu\left(\left(\bigcap_{i=1}^{\infty} B_i\right)\bigcap\left(\bigcap_{i=1}^{\infty} B_i - \frac{3}{4}\right)\right)$$

$= \lim_{i\to\infty} \nu(B_i) - 2\lim_{i\to\infty} \nu\left(B_i \cap\left(B_i - \frac{3}{4}\right)\right)$ based on equation (2)

$$= \lim_{i\to\infty}\left(\nu(B_i) - 2\nu\left(B_i \cap \left(B_i - \frac{3}{4}\right)\right)\right)$$

$$= \lim_{i\to\infty} \mu(B_i).$$

2. Grade-2 additive

To prove the grade-2 additive, then $\mu(E_1 \cup E_2 \cup E_3) = \mu(E_1 \cup E_2) + \mu(E_1 \cup E_3) + \mu(E_2 \cup E_3) - \mu(E_1) - \mu(E_2) - \mu(E_3)$.

Since ν of the Lebesgue measure is additive, hence $\nu(E_1 \cup E_2) = \nu(E_1) + \nu(E_2)$ then it obtained that

$$\mu(E_1 \cup E_2) = \nu(E_1) + \nu(E_2) - 2\nu\left(\left\{x \in E_1 \cup E_2 : x + \frac{3}{4} \in E_1 \cup E_2\right\}\right)$$

$$= \nu(E_1) + \nu(E_2) - 2\nu\left((E_1 \cup E_2)\bigcap\left((E_1 \cup E_2) - \frac{3}{4}\right)\right).$$

Then,

$$\mu(E_1 \cup E_2) + \mu(E_1 \cup E_3) + \mu(E_2 \cup E_3) - \mu(E_1) - \mu(E_2) - \mu(E_3)$$

$$= \nu(E_1) + \nu(E_2) - 2\nu\left((E_1 \cup E_2)\bigcap\left((E_1 \cup E_2) - \frac{3}{4}\right)\right) + \nu(E_1) + \nu(E_3)$$

$$-2\nu\left((E_1 \cup E_3)\bigcap\left((E_1 \cup E_3) - \frac{3}{4}\right)\right) + \nu(E_2) + \nu(E_3)$$

$$-2\nu\left((E_2 \cup E_3)\bigcap\left((E_2 \cup E_3) - \frac{3}{4}\right)\right) - \nu(E_1)$$

$$+2\nu\left(E_1\bigcap\left(E_1 - \frac{3}{4}\right)\right) - \nu(E_2)$$

$$+2\nu\left(E_2\bigcap\left(E_2 - \frac{3}{4}\right)\right) - \nu(E_3) + 2\nu\left(E_3\bigcap\left(E_3 - \frac{3}{4}\right)\right)$$

$$= \nu(E_1) + \nu(E_2) - 2\nu\left((E_1 \cup E_2)\bigcap\left((E_1 \cup E_2) - \frac{3}{4}\right)\right) + \nu(E_1) + \nu(E_3)$$

$$-2\nu\begin{pmatrix}\left(E_1\bigcap\left(E_1 - \frac{3}{4}\right)\right) \cup \left(E_1\bigcap\left(E_3 - \frac{3}{4}\right)\right) \\ \cup \left(E_3\bigcap\left(E_1 - \frac{3}{4}\right)\right) \cup \left(E_3\bigcap\left(E_3 - \frac{3}{4}\right)\right)\end{pmatrix}$$

$$+\nu(E_2) + \nu(E_3)$$

$$-2\nu\begin{pmatrix}\left(E_2\bigcap\left(E_2 - \frac{3}{4}\right)\right) \cup \left(E_2\bigcap\left(E_3 - \frac{3}{4}\right)\right) \\ \cup \left(E_3\bigcap\left(E_2 - \frac{3}{4}\right)\right) \cup \left(E_3\bigcap\left(E_3 - \frac{3}{4}\right)\right)\end{pmatrix} - \nu(E_1) +$$

$$2\nu\left(E_1\bigcap\left(E_1 - \frac{3}{4}\right)\right) - \nu(E_2)$$

$$+2\nu\left(E_2\bigcap\left(E_2 - \frac{3}{4}\right)\right) - \nu(E_3) + 2\nu\left(E_3\bigcap\left(E_3 - \frac{3}{4}\right)\right)$$

$$= \nu(E_1) + \nu(E_2) - 2\nu\left((E_1 \cup E_2)\bigcap\left((E_1 \cup E_2) - \frac{3}{4}\right)\right)$$

$$+\nu(E_1) + \nu(E_3)$$

$$-2\nu\left(E_1\bigcap\left(E_1 - \frac{3}{4}\right)\right) - 2\nu\left(E_1\bigcap\left(E_3 - \frac{3}{4}\right)\right)$$

$$-2\nu\left(E_3\bigcap\left(E_1 - \frac{3}{4}\right)\right)$$

$$-2\nu\left(E_3\bigcap\left(E_3 - \frac{3}{4}\right)\right) + \nu(E_2) + \nu(E_3)$$

$$-2\nu\left(E_2\bigcap\left(E_2 - \frac{3}{4}\right)\right)$$

$$-2\nu\left(E_2\bigcap\left(E_3 - \frac{3}{4}\right)\right) - 2\nu\left(E_3\bigcap\left(E_2 - \frac{3}{4}\right)\right)$$

$$-2\nu\left(E_3\bigcap\left(E_3 - \frac{3}{4}\right)\right) - \nu(E_1)$$

$$+2\nu\left(E_1\bigcap\left(E_1 - \frac{3}{4}\right)\right) - \nu(E_2) + 2\nu\left(E_2\bigcap\left(E_2 - \frac{3}{4}\right)\right)$$

$$-\nu(E_3) + 2\nu\left(E_3\bigcap\left(E_3 - \frac{3}{4}\right)\right) \quad (5)$$

Furthermore,

$$\mu(E_1 \cup E_2 \cup E_3) = \mu((E_1 \cup E_2) \cup E_3)$$

$$= \nu(E_1 \cup E_2) + \nu(E_3) - 2\nu\begin{pmatrix}((E_1 \cup E_2) \cup E_3) \\ \bigcap((E_1 \cup E_2) \cup E_3) - \frac{3}{4}\end{pmatrix}$$

$$= \nu(E_1) + \nu(E_2) + \nu(E_3) - 2\nu\begin{pmatrix}(E_1 \cup E_2)\bigcap \\ \left((E_1 \cup E_2) - \frac{3}{4}\right)\end{pmatrix}$$

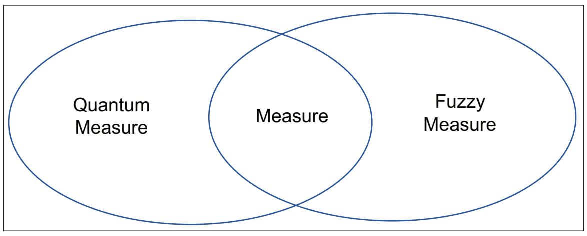

Figure 1: Relationship between measure, fuzzy measure, and quantum measure

related to fuzzy quantum space, convergence, and extreme value analyses, which estimated financial risks using incomplete data. The quantum and fuzzy measures, such as [8], [4], [17], referred to Lattice studied by Birkhoff [20] either directly or indirectly. According to Birkhoff, lattice is a fundamental application of modern algebra, point-set theory, and functional analysis, as well as logic and probability. Furthermore, Gratzer [21] demonstrated that lattice provides a unifying framework for previously unrelated developments in several mathematical disciplines; hence, it is predictable. However, the membership function discussed (point-set) is in a different frame of reference in various studies; hence, they are not in the form of Boolean sigma-algebra.

Although quantum and fuzzy measures are depicted as generalization of measure in several literatures, they are not shown by definition, which, hence, cannot be seen in their generalizations. Furthermore, in various literature, there is also no discussion of the relationship between quantum and fuzzy measures on Boolean σ– algebra. Therefore, this study discusses “the proof of quantum and fuzzy measures as a measure generalization that does not reciprocally generalized.”

Methods

Definition 1 ([22], [23]) *A (Boolean) σ– algebra of sets is a collection S of the subsets from a given set S, such that:*

a. $\phi, S \in S$,

b. *If* $X \in S$ *and* $Y \in S$, *then* $XUY \in S$,

c. *If* $X \in S$, *then* $S - X \in S$,

d. *If* $X_n \in S$, *for all n, then* $\bigcup_{n=0}^{\infty} X_n \in \mathrm{S}$.

Definition 2 ([24]) *The measurable space is a pair of* (*X, M*), *where X is a set and M is an σ algebra of the subset X. Furthermore, the μ in the measurable space* (*X,* M) *is a non-negative function of μ: M* →[0, ∞], *where μ(Ø)* = 0 *and a countably additive in the sense that for any countable disjoint collection* $E_{k(k=1)}^{\infty}$ *of measurable sets satisfy,*

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} E_k.$$

Definition 3 ([24]) *The measure space is triplicate* (*X, M, μ*) *where* (*X, M*) *is the measurable space and μ is the measure within the measurable space* (*X, M*).

Definition 4 ([7], [6]) Let, (X, M) is the *measurable space. A function of* μ: *M* →[0, ∞], *is a quantum measure if:*

lim μ (A_i) = *μ* $(\cup A_i)$, *for every ascending sequence* A_i ∈ *M and* $\lim \mu(B_i) = \mu\left(\bigcap B_i\right)$ *for every descending sequence* B_i ∈ *M (continue),*

1. $\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(A \cup C)$
$+\mu(B \cup C) - \mu(A) - \mu(B) - \mu(C)$
(grade-2 additive)

2. $\mu(A) = 0 \Rightarrow \mu(A \cup B) = \mu(B)$ *and*
$\mu(A \cup B) = 0 \Rightarrow \mu(A) = \mu(B)$ *(regular).*

Definition 5 ([10], [12], [25]) A fuzzy measure *(non-additive measure/capacity)μ in the measurable space* (*X, M*) *is defined as the set of* $\mu: M \rightarrow R^{+}$ *functions, hence:*

1. *lim μ* (A_i) *=μ (U* A_i*), for every ascending sequence* $A_i \in \mathrm{M}$ *dan lim μ* (B_i)*=μ(∩*B_i*), for every descending sequence* B_i∈ *M (continue)*
2. *μ(∅)=0, (Empty)*
3. *μ(A) ≤ μ(B) if A⊂B (monotone)*

Definition 6 ([24]) Suppose E is the set of real numbers, set $\{I_k\}_{k=1}^{\infty}$ as a non-empty open set, and the finite interval covering *E*. The outer measure of *E* is denoted by $v^{*}(E)$ which is defined as follows:

$$v^{*}(E) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) \middle| E \subseteq \bigcup_{k=1}^{\infty} I_k\right\}.$$

Suppose *E* is a measurable set, the Lebesgue measure is denoted by v *(E)* or and defined by $v(E) = v^{*}(E)$.

Theorem 1 ([24]) Measure μ (including *Lebesgue measures) satisfy the following continuity properties:*

1. *If* $\{A_i\}_{i=1}^{\infty}$ *is a collection of ascending measurable sets then:*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i) \qquad (1)$$

$$-2v\left((E_1 \cup E_2)\cap\left(E_3-\frac{3}{4}\right)\right)-2v\left(E_3\cap\left((E_1 \cup E_2)-\frac{3}{4}\right)\right)$$

$$-2v\left(E_3\cap\left(E_3-\frac{3}{4}\right)\right)$$

$$=v(E_1)+v(E_2)+v(E_3)-2v\left((E_1 \cup E_2)\cap\left((E_1 \cup E_2)-\frac{3}{4}\right)\right)$$

$$-2v\left(\left(E_1\cap\left(E_3-\frac{3}{4}\right)\right)\cup\left(E_2\cap\left(E_3-\frac{3}{4}\right)\right)\right)$$

$$-2v\left(\left(E_3\cap\left(E_1-\frac{3}{4}\right)\right)\cup\left(E_3\cap\left(E_2-\frac{3}{4}\right)\right)\right)$$

$$-2v\left(E_3\cap\left(E_3-\frac{3}{4}\right)\right)$$

$$=v(E_1)+v(E_2)+v(E_3)-2v\left((E_1 \cup E_2)\cap\left((E_1 \cup E_2)-\frac{3}{4}\right)\right)$$

$$-2v\left(E_1\cap\left(E_3-\frac{3}{4}\right)\right)-2v\left(E_2\cap\left(E_3-\frac{3}{4}\right)\right)$$

$$-2v\left(E_3\cap\left(E_1-\frac{3}{4}\right)\right)$$

$$-2v\left(E_3\cap\left(E_2-\frac{3}{4}\right)\right)-2v\left(E_3\cap\left(E_3-\frac{3}{4}\right)\right). \qquad (6)$$

According to (5) and (6), $\mu(E_1 \cup E_2 \cup E_3) = \mu(E_1 \cup E_2) + \mu(E_1 \cup E_3) + \mu(E_2 \cup E_3)\ \mu(E_1) - \mu(E_2) - \mu(E_3)$. Therefore, μ is grade-2 additive.

3. Regular

This proves that $\mu(A)=0 \Rightarrow \mu(A \cup B)=\mu(B)$ and $\mu(A \cup B)=0 \Rightarrow \mu(A)=\mu(B)$

First, if $\mu(A)=0 \Leftrightarrow$

$$v(A)-2v\left(A\cap\left(A-\frac{3}{4}\right)\right)=0 \Leftrightarrow v(A)$$

$$=2v\left(A\cap\left(A-\frac{3}{4}\right)\right) \Leftrightarrow v(A)=0$$

or $A=C\cup\left(C-\frac{3}{4}\right)$ for any $C\subseteq\left[\frac{3}{4},1\right]$.

For $v(A) = 0$, then it obtained that

$$\mu(A \cup B)=v(A \cup B)-2v\left((A \cup B)\cap\left(A \cup B-\frac{3}{4}\right)\right)$$

$=v(A) + v(B)$

$$-2v\left(\begin{array}{l}\left(A\cap\left(A-\frac{3}{4}\right)\right)\cup\left(A\cap\left(B-\frac{3}{4}\right)\right)\\ \cup\left(B\cap\left(A-\frac{3}{4}\right)\right)\cup\left(B\cap\left(B-\frac{3}{4}\right)\right)\end{array}\right)$$

$$=v(A)+v(B)-2v\left(A\cap\left(A-\frac{3}{4}\right)\right)-2v\left(A\cap\left(B-\frac{3}{4}\right)\right)$$

$$-2v\left(B\cap\left(A-\frac{3}{4}\right)\right)-2v\left(B\cap\left(B-\frac{3}{4}\right)\right)$$

$$=v(A)+v(B)-2v\left(A\cap\left(A-\frac{3}{4}\right)\right)-2v\left(A\cap\left(B-\frac{3}{4}\right)\right)$$

$$-2v\left(B\cap\left(A-\frac{3}{4}\right)\right)-2v\left(B\cap\left(B-\frac{3}{4}\right)\right)$$

$$=0+v(B)-0-0-0-2v\left(B\cap\left(B-\frac{3}{4}\right)\right)$$

$$=v(B)-2v\left(B\cap\left(B-\frac{3}{4}\right)\right)$$

$$=\mu(B).$$

For $A=C\cup\left(C-\frac{3}{4}\right)$ with any $C\subseteq\left[\frac{3}{4},1\right]$ and it is known that $A\cap B=\emptyset$, hence, the following equations are obtained:

$$C\cap B=\emptyset, \qquad (7)$$

and

$$\left(C-\frac{3}{4}\right)\cap B=\varnothing. \qquad (8)$$

$$\mu(A \cup B)=v(A \cup B)-2v\left((A \cup B)\cap\left((A \cup B)-\frac{3}{4}\right)\right)$$

$$=v(A)+v(B)-2v\left(\begin{array}{l}\left(A\cap\left(A-\frac{3}{4}\right)\right)\cup\left(A\cap\left(B-\frac{3}{4}\right)\right)\\ \cup\left(B\cap\left(A-\frac{3}{4}\right)\right)\cup\left(B\cap\left(B-\frac{3}{4}\right)\right)\end{array}\right)$$

$$=v(A)+v(B)-2v\left(A\cap\left(A-\frac{3}{4}\right)\right)$$

$$-2v\left(\left(A\cap\left(B-\frac{3}{4}\right)\right)\cup\left(B\cap\left(A-\frac{3}{4}\right)\right)\right)$$

$$-2v\left(B\cap\left(B-\frac{3}{4}\right)\right)$$

$$=0+v(B)-2v\left(B\cap\left(B-\frac{3}{4}\right)\right)$$

$$-2v\left(\left(A\cap\left(B-\frac{3}{4}\right)\right)\cup\left(B\cap\left(A-\frac{3}{4}\right)\right)\right)$$

$$=\mu(B)-2v\left(A\cap\left(B-\frac{3}{4}\right)\right)-2v\left(B\cap\left(A-\frac{3}{4}\right)\right)$$

$$= \mu(B) - 2\nu\left(\left(C \cup \left(C - \frac{3}{4}\right)\right) \cap \left(B - \frac{3}{4}\right)\right)$$

$$-2\nu\left(B \cap \left(\left(C \cup \left(C - \frac{3}{4}\right)\right) - \frac{3}{4}\right)\right)$$

$$= \mu(B) - 2\nu\left(\left(C \cap \left(B - \frac{3}{4}\right)\right) \cup \left(\left(C - \frac{3}{4}\right) \cap \left(B - \frac{3}{4}\right)\right)\right)$$

$$-2\nu\left(B \cap \left(\left(C - \frac{3}{4}\right) \cup \left(C - \frac{6}{4}\right)\right)\right)$$

$$= \mu(B) - 2\nu\left(\left(C \cap \left(B - \frac{3}{4}\right)\right) \cup \left((C \cap B) - \frac{3}{4}\right)\right)$$

$$-2\nu\left(B \cap \left(C - \frac{3}{4}\right)\right)$$

$$= \mu(B) - 2\nu\left(\left(C \cap \left(B - \frac{3}{4}\right)\right) \cup \varnothing\right) - 2\nu(\varnothing)$$

(based on equation (7) and (8))

$$= \mu(B) - 2\nu\left(C \cap \left(B - \frac{3}{4}\right)\right)$$

Because $C \subseteq \left[\frac{3}{4}, 1\right]$ and $\left(B - \frac{3}{4}\right) \subseteq \left[0, \frac{1}{4}\right]$, then $C \cap \left(B - \frac{3}{4}\right) = \varnothing$. Therefore, μ ($A \uplus B$)=μ(B).

Second, if μ ($A \uplus B$)=0, then

$$\mu(A \uplus B) = 0 \Leftrightarrow \nu(A \uplus B)$$

$$-2\nu\left((A \uplus B) \cap \left((A \uplus B) - \frac{3}{4}\right)\right) = 0$$

$$\Leftrightarrow \nu(A \uplus B) = 2\nu\left((A \uplus B) \cap \left((A \uplus B) - \frac{3}{4}\right)\right)$$

$$\Leftrightarrow \nu(A) + \nu(B) = 2\nu\left((A \uplus B) \cap \left((A \uplus B) - \frac{3}{4}\right)\right)$$

$\Leftrightarrow A \uplus B=\emptyset$ or $A \uplus B = C \uplus \left(C - \frac{3}{4}\right)$ for any $C \subseteq \left[\frac{3}{4}, 1\right]$.

For $A \uplus B=\emptyset$, then $A=\emptyset$ and $B=\emptyset$, therefore μ (A) = μ(B). Furthermore, for $A \uplus B = C \uplus \left(C - \frac{3}{4}\right)$ for any $C \subseteq \left[\frac{3}{4}, 1\right]$, then

$$\mu(A \uplus B) = 0 \Leftrightarrow \nu(A) + \nu(B)$$

$$-2\nu\left((A \uplus B) \cap \left((A \uplus B) - \frac{3}{4}\right)\right) = 0$$

$$\Leftrightarrow \nu(A) + \nu(B) - 2\nu\left(A \cap \left(A - \frac{3}{4}\right)\right)$$

$$-2\nu\left(\left(A \cap \left(B - \frac{3}{4}\right)\right) \cup \left(B \cap \left(A - \frac{3}{4}\right)\right)\right)$$

$$-2\nu\left(B \cap \left(B - \frac{3}{4}\right)\right) = 0$$

$$\Leftrightarrow \mu(A) + \mu(B) - 2\nu\left(A \cap \left(B - \frac{3}{4}\right)\right) - 2\nu\left(B \cap \left(A - \frac{3}{4}\right)\right) = 0.$$

Because μ ($A \uplus B$)=0, then $\nu\left((A \uplus B) \cap \left(\frac{1}{4}, \frac{3}{4}\right)\right) = 0$ and $A \cap B=\emptyset$. According to equation (9),

$$\mu(A) = \nu\left(A \cap \left(B - \frac{3}{4}\right)\right) + \nu\left(\left(A - \frac{3}{4}\right) \cap B\right)$$

$$= \nu\left(\left(B - \frac{3}{4}\right) \cap A\right) + \nu\left(B \cap \left(A - \frac{3}{4}\right)\right) = \mu(B).$$

This showed that (X, M, μ) is a quantum measure space.

4. Not Fuzzy Measure

$\mu\left(\left[0, \frac{1}{4}\right] \uplus \left[\frac{3}{4}, 1\right]\right) = 0$ meanwhile, $\mu\left(\left[0, \frac{1}{4}\right]\right) = \frac{1}{4}$, the monotony is not satisfied and μ is not a fuzzy measure.

Therefore, the quantum measure is not a generalized fuzzy measure.

Proof that fuzzy measure is not a quantum measure generalization

This is conducted with the following counter example. For example, $X = \{a,b\}$ and M of algebra -σ from X, as well as define the μ measure, namely:

$$\mu(E) = \begin{cases} 1; E = X \\ 0; E \neq X \end{cases} \text{with} E \in \mathrm{M} \qquad (11)$$

It is shown that (X, M, μ) is a fuzzy measure space, where μ is continuous (1), empty (2) and $\mu(A) \leq \mu(B)$ if $A \subset B$ (3), though not a quantum measure space (4).

1. Continuous

When X is finite then μ satisfies the definition of continuous on the fuzzy measure.

2. Empty

Because $\emptyset \neq \{a,b\} = X$ then satisfies the definition of $\mu(\emptyset) = 0$.

3. Monotone

When $\emptyset \neq X$, $\{a\} \neq X$, $\{b\} \neq X$ then $\mu(\emptyset)=\mu(\{a\})=\mu(\{b\})=0$ but $\mu(X)=1$, hence

$\emptyset \subset \{a\}$ and apply $\mu(\emptyset) = 0 \leq 0 = \mu(\{a\})$,

∅⊂{b} and app ly $\mu(\emptyset) = 0 \leq 0 = \mu(\{b\})$,

∅⊂X and apply $\mu(\emptyset) = 0 \leq 1 = \mu(X)$,

{a}⊂X and apply $\mu(\{a\}) = 0 \leq 1 = \mu(X)$,

{b}⊂X and apply $\mu(\{b\})= 0 \leq 1=\mu(X)$

This depicts that (X, M, μ) is a fuzzy measure space.

4. Not Quantum Measure

$\mu(\{a\})=0$ but $\mu(\{a\} \uplus \{b\}) = \mu(\{a,b\}) = 1 \neq 0 = \mu(\{b\})$ This does not fulfill the regular properties, and the result is not a quantum measure.

Therefore, the fuzzy measure is not a generalization of the quantum measure.

Figure 1 illustrates the results obtained based on the above discussion.

Furthermore, a comparison of calculations will be carried out. Suppose $X = [0,1]$, M is the σ- algebra of X, and v is the Lebesgue measure constrained to [0,1] Therefore $E \in M$, μ^* and $\mu^\#$ are defined as:

$$\mu^*(E) = \nu(E) - 2\nu\left(\left\{x \in E : x + \frac{3}{4} \in E\right\}\right)$$

$$= \nu(E) - 2\nu\left(E \bigcap \left(E - \frac{3}{4}\right)\right)$$

And $$\mu^\#(E) = \begin{cases} \nu(E), if\, \nu(E) \leq \dfrac{1}{2} \\ \dfrac{1}{2}, else \end{cases}$$

Subsection 3.2 and 3.3 showed that (X, M, μ^*) is a quantum measure space but not a generalized fuzzy measure, and $(X, M, \mu^\#)$ is a fuzzy measure space but not a generalized quantum measure. Table 1 depicts the comparison between the calculation of measure, quantum measure, and fuzzy measure in a measurable space (X,M).

Discussion

Quantum and fuzzy measures appear separately, where the quantum measure starts from the need for measuring instruments in physical phenomena, while the fuzzy size arises from phenomena with fuzzy occurrences. It has been shown that although both quantum and fuzzy measures generalize to measure, they do not generalize to each other. Table 1 shows an example of a comparison of quantum and fuzzy measure calculations, where the results obtained differ significantly. This comparison is not intended to obtain a better measure, because each measure has its own application. However, acquiring a measure that can contain quantum and fuzzy measures is certainly needed to see causal relationships and arrange the layout between measure theories as well as add to the scientific repertoire of measure theory that can be applied to quantum physics phenomena and various vague events.

Table 1: Comparison between the calculation of measure, quantum measure, fuzzy measure

Serial number	Interval (E)	Measure (υ)	Quantum measure (μ^*)	Fuzzy measure ($\mu^\#$)
1	(0,1)	1	0.5	0.5
2	(0, 0.75)	0.75	0.75	0.5
3	(0.75, 1)	0.25	0.25	0.25
4	(0.25, 0.75)	0.5	0.5	0.5
5	(0, 0.25) U (0.75, 1)	0.5	0	0.5
6	(0, 0.25) U (0.5, 1)	0.75	0.25	0.5
7	(0, 0.5) U (0.75, 1)	0.75	0.25	0.5
8	(0, 0.1) U (0.25, 0.75) U (0.9, 1)	0.7	0.7	0.5
9	(0.2, 0.9)	0.7	0.7	0.5
10	(0.25, 1)	0.75	0.75	0.5
11	(0.1, 0.95)	0.85	0.65	0.5
12	(0.1, 0.2) U (0.85, 0.95)	0.2	0	0.2
13	(0.2, 0.25) U (0.95, 1)	0.1	0	0.1
14	(0.2, 0.3) U (0.7, 0.8)	0.2	0.1	0.2
15	(0, 0.1) U (0.4, 0.6) U (0.75, 0.85)	0.4	0.2	0.4

Conclusions

This study discusses the proof of quantum and fuzzy measures as a measure generalization on Boolean σ- algebra, and the conclusions obtained are as follows:

1. Quantum and fuzzy measures are measure generalizations
2. Quantum measure is not a generalization of the fuzzy measure
3. Fuzzy measure is not a generalization of the quantum measure.

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